

REGULAR FACTORS IN NEARLY REGULAR GRAPHS

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We strengthen classical theorems of Petersen, Ore, B  bler, Gallai on the existence of regular factors in regular graphs to nearly regular graphs.

1. Introduction

An h -factor of a graph G , h an integer, is a spanning subgraph which is regular of degree h . More generally, given an integer-valued function f defined on $V(G)$, an f -factor is a subgraph H of G such that $d_H(x) = f(x)$ for all $x \in V(G)$. Several important theorems of the theory of graph factors give simple sufficient conditions for the existence of h -factors in regular graphs [1, 2, 5].

Theorem (Petersen [5]). *A regular graph of even degree k can be decomposed into $\frac{1}{2}k$ 2-factors.*

Theorem (Ore [4, Theorem 3.2.5]). *Let G be a 2-edge-connected regular graph of odd degree k and h be an even integer $2 \leq h \leq 2k/3$. Then G contains an h -factor.*

Theorem (k odd: B  bler [1]; k even: Gallai [2]). *Let G be a connected regular graph of degree k , with an even number of vertices. Let h be an odd integer, $h \leq \frac{1}{2}k$. If for all $X \subseteq V$, $|X|$ odd we have $e(X, V - X) \geq k/h$, then G has an h -factor.*

In the present note we give short proofs of these theorems by deriving them from Tutte's f -Factor Theorem [7]. We obtain slight generalizations: Due to integrality conditions these existence theorems remain valid for graphs which are almost regular (in a sense to be made precise below).

For completeness we recall Tutte's f -Factor Theorem [7].

* C.N.R.S.

Theorem. *A graph has an f -factor if and only if*

$$f(X) + d_G(Y) - f(Y) \geq e_G(X, Y) + \left(\begin{array}{l} \text{number of connected components} \\ C \text{ of } G[V \setminus (X \cup Y)] \text{ such that} \\ e_G(C, Y) \not\equiv f(C) \pmod{2} \end{array} \right)$$

for all $X, Y \subseteq V(G)$ with $X \cap Y = \emptyset$.

The reader is referred to [3] for a short proof of the 1-Factor theorem [6], and to [8] for a short proof of the f -Factor Theorem from the 1-Factor theorem. Another necessary and sufficient condition for the existence of f -factors is given by Ore [4].

2. Notations

Graphs considered in the present note are finite, undirected, and may have loops and multiple edges (a loop incident to a vertex x contributes 2 to the degree of x).

Throughout the paper the following notations are used: Given a graph G the vertex-set of G is denoted by $V(G)$. The degree in G of a vertex x is $d_G(x)$. Given $X \subseteq V(G)$, we set $d_G(X) = \sum_{x \in X} d_G(x)$. the subgraph of G induced by X is denoted by $G[X]$, the number of edges of $G[X]$ by $e_G(X)$. Given $X, Y \subseteq V(G)$ with $X \cap Y = \emptyset$ we denote by $e_G(X, Y)$ the number of edges of G joining X and Y .

If no confusion results subscripts G will usually be omitted.

3. The theorems

Theorem 1 (even factors). *Let G be a graph and h, k be two integers such that h is even and $2 \leq h \leq \frac{2}{3}k$. Suppose that $\sum_{x \in V(G)} |d_G(x) - k| < k/h$ and G is 2-edge connected. Then G contains an h -factor.*

Proof. Tutte's condition for the existence of an h -factor, h even, is:

$$h |X| + d(Y) - f |Y| \geq e(X, Y) + \left(\begin{array}{l} \text{number of connected components } C \\ \text{of } G[V \setminus (X \cup Y)] \text{ such that} \\ e(C, Y) \text{ is odd} \end{array} \right)$$

for all $X, Y \subseteq V$ with $X \cap Y = \emptyset$.

The condition is clearly satisfied for $X = Y = \emptyset$. Let $X, Y \subseteq V$ with $X \cap Y = \emptyset$ and $X \cup Y \neq \emptyset$. We distinguish two types of connected components C of

$G[V \setminus (X \cup Y)]$ such that $e(C, Y)$ is odd:

(1) $e(C, X) = 0$. Since G is 2-edge-connected $e(C, Y) = e(C, V \setminus C) \geq 2$. Hence, $e(C, Y)$ being odd, we have $e(C, Y) \geq 3$.

(2) $e(C, X) \geq 1$. Since $e(C, Y)$ is odd, we have $e(C, Y) \geq 1$.

Let p_1, p_2 be the numbers of components of each type. We have clearly

$$\begin{aligned} p_2 + e(X, Y) &\leq e(X, V \setminus X) \leq d(X), \\ 3p_1 + p_2 + e(X, Y) &\leq e(Y, V \setminus Y) \leq d(Y). \end{aligned}$$

Hence the inequalities

$$\begin{aligned} p_2 + e(X, Y) &\leq d(X), \\ 3p_1 + p_2 + e(X, Y) &\leq d(Y). \end{aligned}$$

We multiply the first inequality by h/k , the second one by $(k-h)/k$ and add. We obtain

$$3 \frac{k-h}{k} p_1 + p_2 + e(X, Y) \leq \frac{h}{k} d(X) + \frac{k-h}{k} d(Y).$$

Since $3(k-h)/k \geq 1$ if $h \leq \frac{2}{3}k$ it follows, with $p = p_1 + p_2$, that

$$p + e(X, Y) \leq \frac{h}{k} d(X) + \frac{k-h}{k} d(Y).$$

Suppose

$$h |X| + d(Y) - h |Y| < p + e(X, Y).$$

Since both sides are integers we have

$$h |X| + d(Y) - h |Y| + 1 \leq p + e(X, Y) \leq \frac{h}{k} d(X) + \frac{k-h}{k} d(Y).$$

It follows that

$$d(X) - k |X| - (d(Y) - k |Y|) \geq \frac{k}{h}.$$

Since $X \cap Y = \emptyset$ we have

$$d(X) - k |X| - (d(Y) - k |Y|) \leq \sum_{x \in X \cup Y} |d(x) - k| < \frac{k}{h},$$

a contradiction. Hence Tutte's condition is satisfied. Theorem 1 follows. \square

The theorems of Petersen and Ore quoted in the introduction correspond to the particular cases of Theorem 1 when G is regular. To obtain Petersen's theorem it suffices to observe that the connected components of a regular graph of even degree are necessarily 2-edge-connected. (Proof: for $X \subseteq V$ we have $k |X| = d(X) = 2e(X) + e(X, V \setminus X)$, hence if k is even $e(X, V \setminus X)$ is also even.)

Theorem 2 (odd factors). *Let G be a connected graph with an even number of vertices, and h, k be two integers such that h is odd and $1 \leq h \leq \frac{1}{2}k$. Suppose that $\sum_{x \in V(G)} |d(x) - k| < k/h$ and for all $X \subseteq V$ with $e(X, V - X)$ odd or $|X|$ odd we have $e(X, V - X) \geq k/h$. Then G contains an h -factor.*

Note. The condition on $e(X, V - X)$ is obviously satisfied if the edge connectivity of G is at least k/h .

Proof. Tutte's condition for the existence of an h -factor, h odd, is:

$$h |X| + d(Y) - h |Y| \geq e(X, Y) + \left(\begin{array}{l} \text{number of connected components } C \\ \text{of } G[V \setminus (X \cup Y)] \text{ such that} \\ e(C, Y) \not\equiv |C| \pmod{2} \end{array} \right) \quad |$$

for all $X, Y \subseteq V$ with $X \cap Y = \emptyset$.

The condition is clearly satisfied for $X = Y = \emptyset$. Let $X, Y \subseteq V$ with $X \cap Y = \emptyset$ and $X \cup Y \neq \emptyset$. We distinguish four types of connected components C of $G[V \setminus (X \cup Y)]$ such that $e(C, Y) \not\equiv |C| \pmod{2}$.

(1) $|C|$ even, $e(C, X) = 0$. Since $e(C, Y)$ is odd we have $e(C, Y) = e(C, V \setminus C) \geq k/h$.

(2) $|C|$ even, $e(C, X) \geq 1$. Since $e(C, Y)$ is odd we have $e(C, Y) \geq 1$.

(3) $|C|$ odd, $e(C, Y) = 0$. We have $e(C, X) = e(C, V \setminus C) \geq k/h$ by hypothesis.

(4) $|C|$ odd, $e(C, Y) \geq 1$. Since $e(C, Y)$ is even we have $e(C, Y) \geq 2$.

Let p_1, p_2, p_3, p_4 be the number of components of each type. We have clearly

$$p_2 + \frac{k}{h} p_3 + e(X, Y) \leq e(X, V \setminus X) \leq d(X),$$

$$\frac{k}{h} p_1 + p_2 + 2p_4 + e(X, Y) \leq e(Y, V \setminus Y) \leq d(Y).$$

Hence the inequalities

$$p_2 + \frac{k}{h} p_3 + e(X, Y) \leq d(X),$$

$$\frac{k}{h} p_1 + p_2 + 2p_4 + e(X, Y) \leq d(Y).$$

We multiply the first inequality by h/k , the second one by $(k - h)/k$, and add. We obtain

$$\frac{k - h}{h} p_1 + p_2 + p_3 + 2 \frac{k - h}{k} p_4 + e(X, Y) \leq \frac{h}{k} d(X) + \frac{k - h}{k} d(Y).$$

Since $(k - h)/h \geq 1$ and $2(k - h)/k \geq 1$ if $h \leq \frac{1}{2}k$, it follows with $p = p_1 + p_2 + p_3 + p_4$, that

$$p + e(X, Y) \leq \frac{h}{k} d(X) + \frac{k - h}{k} d(Y).$$

As in the proof of Theorem 1 this inequality implies that Tutte's condition is satisfied. \square

The theorem of Babler and Gallai corresponds to the particular cases of Theorem 2 when G is regular. If G is regular of degree k we have $k|X| = d(X) = 2e(X) + e(X, V - X)$. Hence if k is odd, $e(X, V - X)$ is odd if and only if $|X|$ is odd and if k is even, $e(X, V - X)$ is always even. Therefore the hypothesis of Theorem 2 reduces to: for all $X \subseteq V$, $|X|$ odd implies $e(X, V - X) \geq k/h$.

Finally, we note that the bounds on h and $\sum_{x \in V} |d(x) - k|$ in Theorems 1 and 2 are probably tight. (We leave as an exercise to the reader the construction of effective examples).

References

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